4. Given a coin with probability p of landing on heads after a flip, what is the probability that the number of heads will ever equal the number of tails assuming an infinite number of flips?

Answer:

This is a standard random walk problem. Consider the starting point to be 0, getting a head to be moving one step to the right, and getting a tail to be moving one step to the left. Then the probably to the right is p, and the probability of moving to the left is 1-p, and the probability of the number of heads ever equals the number of tails is essentially the probability that moving back to the origin after the first step

(a) The direction of positive and negative is arbitrary here, so the probability of p and 1-p is symmetric. Let ‘s consider the case when p >= 0.5, and the situation when p < 0.5 would be similar.

Let’s call:

A = probability of ever moving one step to the right of where you are

B = probability of moving back to the current position from its left

For A, to ever move one step to the right, one can move one step to its right directly, or move to the left and then moving back, or even back and forth like this many times, so

A = p + Bp + B2p + B3p + ……

When we flip the coin for infinite times,

A = p /(1-B)

Note that A is a probability distribution, so p/(1-B) <= 1, so we know B <= 1-p here

Also for B, the probability of moving back to the current position from the left is equal to the probability of moving one step to the left, times the probability of moving back to the current position from either direction is

B = (1-p)A

Solve for B, and we have:

B/(1-p) = p/(1-B), B = p or 1-p

When p > 0.5, B <= 1-p < 0.5, so B = p

When p =0.5, B = 0.5, also B = p

So the probability of moving back to the current position from its left is p. Because of the symmetry, we know that the probability of moving back to the current position from its right is also B = p. So the probability of moving back to the current position is 2p.

(b) When p < 0.5, repeat the above process by replacing p with 1-p. So the probability of moving back to the current position is 2(1-p).

Combine the results of the two cases, and the number of heads will ever equal the total number of tails p = 2min(p, (1-p))